

Spreading and localization of wavepackets in disordered wires in a magnetic field

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We study the diffusive and localization properties of wavepackets in disordered wires in a magnetic field. In contrast to a recent supersymmetry approach our numerical results show that the decay rate of the steady state changes *smoothly* at the crossover from preserved to broken time-reversal symmetry. Scaling and fluctuation properties are also analyzed and a formula, which was derived analytically only in the pure symmetry cases is shown to describe also the steady state wavefunction at the crossover regime. Finally, we present a scaling for the variance of the packet which shows again a smooth transition due to the magnetic field.

In recent years, considerable progress has been made in understanding the structure of eigenstates of quasi-one-dimensional (quasi-1D) disordered systems (see [1] and references therein). Most of the investigations have been concerned with the cases of completely preserved and totally broken time-reversal symmetry. The transition between these limits, however, has been studied much less and little is known about it. The only results for the transition were based on a heuristic approach by Bouchaud [2], a semiclassical analysis by Imry and Lerner [3] and numerical studies based on transfer matrices by Pichard et al. [4]. The main prediction of the above studies is the doubling of the localization length in the limit of strong magnetic fields while a smooth transition towards this asymptotic limit was assumed. Similar results were obtained for quantum chaotic models [5]. Recently, the doubling of the localization length was observed in sub-micron thin wires of doped GaAs [6], where for increasing magnetic field strengths a continuously decreasing activation energy was reported, which saturated indeed at half of its field free value. Motivated by this experiment, the first attempt to study such a transition using the supersymmetry technique was undertaken in [7,8]. These authors predict that the decay of wavefunctions in disordered wires in weak magnetic fields, is characterized by two localization lengths; the far tails decay with the length l_{tails} characteristic for completely broken time-reversal symmetry, while at shorter distances the decay length is $l_{\text{orig}} = l_{\text{tails}}/2$.

The predictions of [7,8] imply two different temperature regimes in the hopping conductivity separated by a boundary which depends on the magnetic field strength. A direct numerical test of the above prediction by means of the transmission approach was carried out recently in [9]. The results are contradicting the two-scale behaviour of the wavefunction in the crossover regime. The following possibilities were proposed in order to explain this discrepancy [7–9]: (a) the Borland conjecture (which connects the asymptotic decay length of the transmittance to the asymptotic decay length of the wavefunc-

tion) breaks down in the crossover regime between the preserved and broken time reversal symmetry, and (b) the two-scale localization phenomenon is due to anomalously localized states that are irrelevant for a typical wire. The latter case appears more likely since in [7,8] the average wavefunction was calculated whereas, in [9] under the assumption of the Borland conjecture the authors calculated the average of the logarithm of the wavefunction. We are going to show that none of the above reasons can resolve the contradictions!

In this Letter the problem of the crossover behaviour, from preserved to broken time reversal symmetry, is addressed from a totally different perspective. In particular we study the evolution of initially δ -like wavepackets in a quasi-1D geometry under the influence of a magnetic field. We then directly calculate the logarithmic and the arithmetic average of the steady state wavefunction. We show that the extracted decay rates change smoothly as a function of the magnetic field for both observables. Moreover, we study the whole steady state distribution and its fluctuations as a function of the magnetic field. From our numerical data we extract an analytical formula for the asymptotic wavefunction in the crossover regime. Finally, we present arguments according to which the diffusive constant scales smoothly with respect to the magnetic field. This imposes a smooth scaling behaviour of the variance of the packet. Our predictions are confirmed by extensive numerical calculations.

The mathematical model we consider is the time-dependent Schrödinger equation on a 1D lattice,

$$i \frac{dc_n(t)}{dt} = \sum_{m=n-b}^{n+b} H_{nm} c_m \quad , \quad (1)$$

where $c_n(t)$ is the probability amplitude for an electron to be at site n and $H_{nm} = H_{nm}^0 + i\alpha A_{nm}$ is a complex Hermitian Band Random Matrix (BRM), which is decomposed into a real symmetric matrix H^0 and a real antisymmetric matrix A with imaginary weight $i\alpha$. The entries of the two matrices are independent Gaussian random numbers with variance $\sigma^2 = 1 + \delta_{nm}$ (where δ_{nm}

is the Kronecker symbol) if $|n - m| \leq b$ and zero otherwise. The parameter b defines the hopping range between neighbouring sites, or, in the quasi-1D interpretation, the number of transverse channels along a thin wire. To relate the parameter α to the magnetic field B , we note that the perturbation of the levels is proportional to the magnetic flux $\Phi = kBl_\infty$ through an area limited by the localization length. Here k is a dimensionless constant of order unity that depends on the specific geometry of the disordered wire [8,10]. Hence we expect $\alpha \sim \Phi/\Phi_0$, where $\Phi_0 = h/e$ is the elementary flux quantum.

We have integrated Eq. (1) numerically using a Cayley scheme in order to preserve the norm [11]. Moreover a self-expanding algorithm was implemented to eliminate finite-size effects. Whenever the probability of finding the particle at the edges of the lattice exceeded 10^{-15} , $20b$ new sites were added to each end. Since all eigenstates of the Hamiltonian H_{nm} are known to be exponentially localized with a localization length $l_\infty(E) \sim b^2$ [13], the evolution of the wavepacket is expected to exhibit a relaxation to a steady-state distribution in the limit $t \rightarrow \infty$. On the basis of numerical calculations it was shown in [12] that for the time reversal symmetry the asymptotic profile $f_s(n) \equiv |c_n(t \rightarrow \infty)|^2$ is given by the following expression:

$$f_s(x) = \frac{\pi^2}{16l_\infty} \int_0^\infty \frac{\eta \text{sh}(\pi\eta)(1+\eta^2)^2}{(1+\text{ch}(\pi\eta))^2} e^{-\frac{1+\eta^2}{4l_\infty}|x|} d\eta, \quad (2)$$

where l_∞ is the averaged (over energy) localization length. Equation (2) was later proved analytically, for quasi-1D systems with preserved and broken time reversal symmetry [14]. It is interesting to note that Eq. (2) was derived first for continuous 1D models with white noise potential [15] with l_∞ being the mean free path. Remarkably, in spite of the relevant difference between the 1D and quasi-1D case (the mean free path is of the order of b in the latter and thus much smaller than the localization length $l_\infty \sim b^2$), the asymptotic shape remains the same in both cases. From (2), one finds that close to the origin,

$$f_s(x) \sim \exp(-|x|/l_\infty); \quad |x| \leq l_\infty, \quad (3)$$

while the asymptotic decay is described by

$$f_s(x) \sim |x|^{-3/2} \exp(-|x|/4l_\infty); \quad |x| \gg 4l_\infty, \quad (4)$$

revealing that the decay rate $s(x) = d \log f_s / dx$ changes by a factor 4.

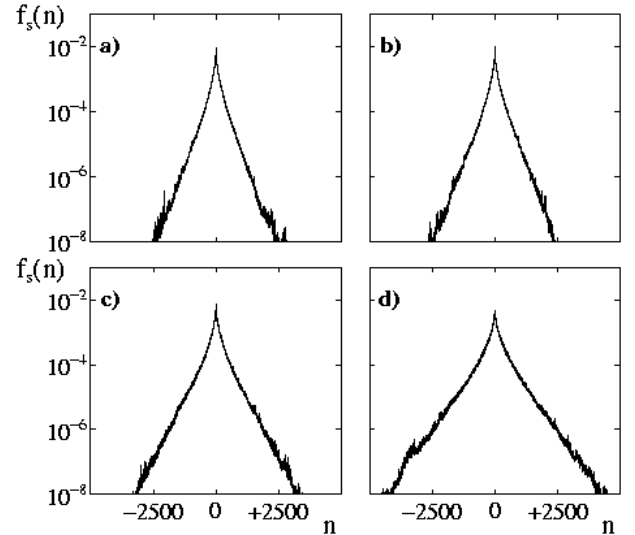


FIG. 1. Arithmetic average of the asymptotic wavefunction $f_s(n)$. The bandwidth is $b = 15$, and the snapshot is taken at $t = 10^4$. The data correspond to various crossover parameters: (a) $\alpha = 0$; (b) $\alpha = 0.02$; (c) $\alpha = 0.085$; and (d) $\alpha = 0.7$.

In Fig. 1 we report some asymptotic profiles $f_s(x, \alpha)$ for $b = 15$ and various values of the time-reversal symmetry breaking parameter α . These data strongly suggest that the decay of $f_s(n)$ in the vicinity of the origin is definitely faster than in the tails. As no analytical results are available for the crossover regime, it is very tempting to compare our numerical results with the theoretical dependence (2) derived for quasi-1D systems with pure symmetries. The profiles reported in Fig. 2a are the result of an arithmetic average over many realizations of the asymptotic profile for $b = 5, 10, 15$ and several α . They are plotted with the scaling assumption

$$f_s(n, t \rightarrow \infty) = l_\infty f_s(x); \quad x \equiv n/l_\infty \quad (5)$$

where l_∞ is determined by a fit according to Eq. (2). The very good agreement between the numerical results and the analytical curve over a broad range of x -values suggests that a properly modified theory including the effect of an intermediate magnetic field should be able to account for the asymptotic profile of wavepackets in quasi-1D systems. Furthermore the scaling relation (5) implies that the wavefunction in the crossover regime shows the same gross structure (envelope) as in the pure symmetry cases, on scales comparable to the localization length. This is in contrast to the occurrence of a second localization scale for the far tails proposed in [7,8]. Hence we expect that all the scaling laws for the eigenstates dominated by the fluctuations of the "envelope" (e.g., moments like the inverse participation ratio), also hold for the crossover regime.

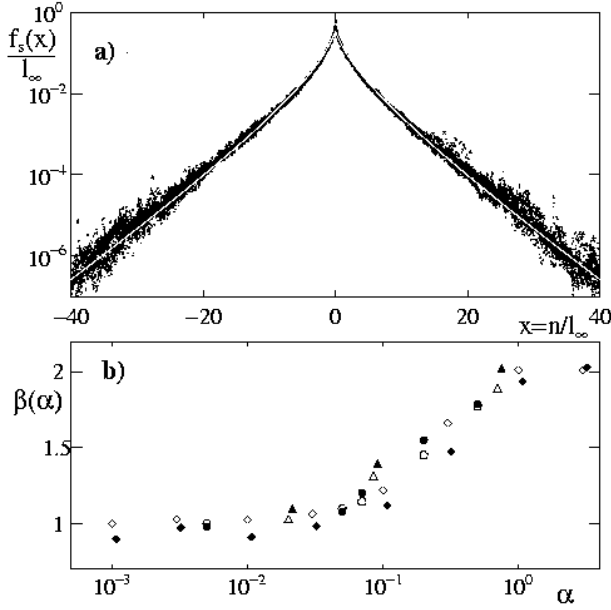


FIG. 2. (a) Asymptotic average profile of the wavepackets for $b = 5, 10, 15$ and various crossover parameters α rescaled according to Eq. (5). The light smooth line is the theoretical expression (2). (b) The localization lengths l_∞ (open symbols) and l_∞^{as} (full symbols) as a function of the time reversal symmetry breaking parameter α . Various symbols correspond to various values of b ; diamonds to $b = 5$, circles to $b = 10$ and triangles to $b = 15$.

In Fig. 2b the asymptotic localization length l_∞ , as determined from the best fit with (2) is plotted versus the crossover parameter α . One can clearly see that the transition from preserved to broken time-reversal symmetry is rather smooth and thus we have

$$l_\infty(\alpha) = \beta(\alpha)l_\infty(0) \quad (6)$$

where $\beta(\alpha)$ is a smooth function that interpolates between the values 1 and 2 for preserved ($\alpha = 0$) and totally broken time reversal symmetry ($\alpha \simeq 1$) respectively. In Fig. 2b we also present the localization length derived by a direct fit of the average of the logarithm of the asymptotic wavefunction $l_\infty^{\text{as}} = \lim_{N \rightarrow \infty} \langle \ln(f_s) \rangle / (2N)$. It clearly shows the same smooth behaviour as l_∞ .

Our numerical results are in contrast to the supersymmetry results [7,8] and agree nicely with the transfer matrix calculations [9]. Moreover, they give a definite answer to the question, whether the assumed discrepancy is due to the fact that in [7,8] the logarithm of the averaged wavefunction was investigated, while in [9] it was the logarithmic average. Based on our calculations we are now able to exclude both possibilities suggested in [8,9]; namely that one could attribute the two-scale localization phenomenon to anomalously localized states, that dominate the asymptotic average profile $f_s(n)$ or to the non-applicability of the Borland conjecture in the crossover regime.

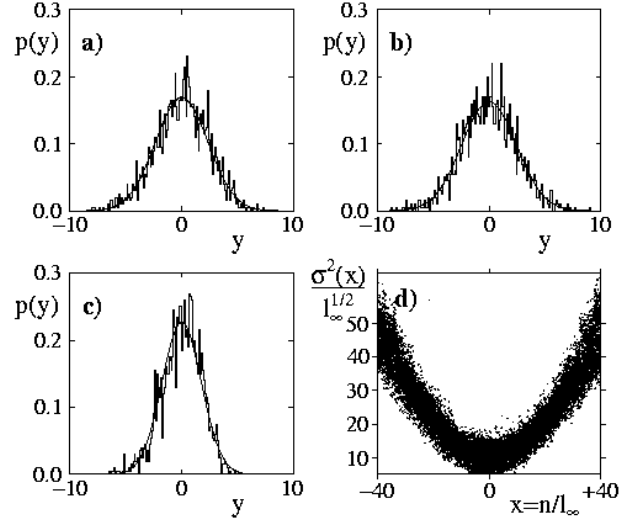


FIG. 3. Distribution of $y \equiv \ln f_s(n = 6b^2 \pm 2)$ for $b = 5$ and various values for the time-reversal symmetry breaking parameter: (a) $\alpha = 0.003$; (b) $\alpha = 0.03$; (c) $\alpha = 0.3$; (d) Variance of y for $b = 5, 10, 15$ and various α . The data are rescaled according to Eq. (8).

As mentioned above, the localization of all eigenstates implies that for $t \rightarrow \infty$ the quantum steady state $f_s(n)$ is localized and fluctuates around the average profile (2). Thus, one can ask about the distribution of $f_s(n)$. Having in mind that the asymptotic profile is exponentially localized (see Eq. (2)), we computed the distribution of the logarithm of the wavefunction $y \equiv \ln(f_s)$ for various values of the crossover parameter α . In Figs. 3a-c we report some representative distributions $\mathcal{P}(y)$, which refer to the case $b = 5$ with $\alpha = 0.003, 0.03$, and $\alpha = 0.3$. In all cases, we have found that the distribution of the logarithm of the wavefunction $\mathcal{P}(y)$, is a Gaussian with good accuracy. This is in perfect agreement with the analytical calculation of [8] for $\mathcal{P}(y)$. In particular a useful indicator is, the spatial growth of the asymptotic variance

$$\sigma^2(n) = \langle (\ln f_s(n))^2 \rangle - \langle \ln f_s(n) \rangle^2. \quad (7)$$

The results for the cases $b = 5, 10$ and various values $0 \leq \alpha \leq 1$ are reported in Fig. 3d under the scaling assumption

$$\sigma^2(n) = \sigma^2(x) / \sqrt{l_\infty}; \quad n/l_\infty. \quad (8)$$

This further confirms that not only the mean asymptotic profile $f_s(n)$ but also higher moments change smoothly as a function of the crossover parameter α .

A global characterization of the diffusion and localization properties of a wavepacket is provided by the evolution of the mean-square displacement

$$M(t, \alpha) \equiv \left\langle \sum_m m^2 |c_m(t)|^2 \right\rangle, \quad (9)$$

where $\langle \cdot \rangle$ denotes the average over different realizations of the disorder. On the basis of the localization properties of the asymptotic wavefunction (see Eqs. (2,6)) one expects that, for $b \gg 1$, M_∞ grows as $M_\infty \sim l_\infty(\alpha)^2$. The quantity M reaches its maximum value M_∞ at $t \simeq t_D$. Up to that time the evolution of the packet is diffusive, i.e. $M_\infty \simeq M(t_D) \simeq Dt_D$, where D is the diffusion constant. From supersymmetry [10] it is known that $l_\infty \Delta = D$ with Δ being the mean level spacing. The latter depends on α as $\Delta(\alpha) = \sqrt{1 + 0.5\alpha^2} \Delta(0)$. Accordingly the following relations hold:

$$\frac{t_D(\alpha)}{t_D(0)} = \frac{\beta(\alpha)}{\sqrt{1 + 0.5\alpha^2}}; \quad \frac{D(\alpha)}{D(0)} = \beta(\alpha) \sqrt{1 + 0.5\alpha^2} \quad (10)$$

where $t_D(0) \sim b^{3/2}$ and $D(0) \sim b^{5/2}$ [12]. In the inset of Fig. 4 we show the scaling of the diffusion constant $D(\alpha)$ for $b = 30, 40$ and various α . Our numerical results are in perfect agreement with (10). Equation (10) suggests that the mean-square displacement $M(t)$ follows the scaling relation

$$M(t, \alpha) = M_\infty(\alpha) \tilde{M}(t/t_D(\alpha)). \quad (11)$$

The numerical results obtained for the cases $b = 5, 10, 15$ and various values of the crossover parameter α are reported in Fig. 4 according to the above ansatz. The close coincidence of data reveals the presence of a scaling regime already at moderately large b -values.

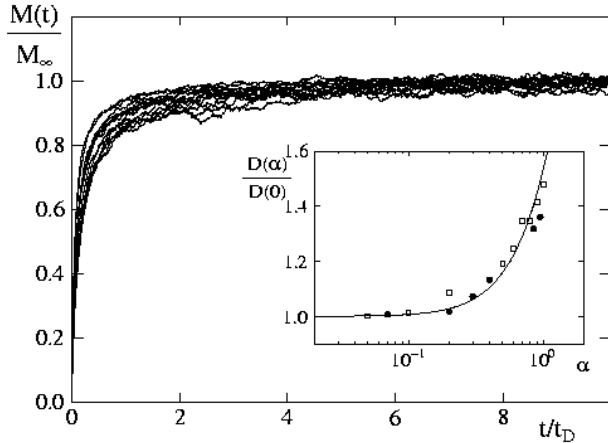


FIG. 4. Mean-square displacement $M(t)$ for $b = 5, 10, 15$ and various values of α , confirming the scaling law (11). In the inset we present the diffusion constant $D(\alpha)$ as a function of the time-reversal symmetry breaking parameter α for $b = 30$ (squares), 40 (filled circles). The solid line is the theoretical expectation (10).

In conclusion, we have studied scaling properties of both diffusion and strong localization of wavepackets in quasi-1D (i.e., in 1D random media with long-range interactions). Our numerical data show that the decay rate of

the asymptotic wavefunction profile changes smoothly, in contradiction to the supersymmetry results [7,8]. We have also found that the asymptotic shape of the wavepacket is well reproduced by the analytical expression (2) derived for the pure symmetry cases (i.e. $\alpha = 0, 1$). Moreover, in agreement with [8] we find, that the logarithm of the asymptotic wavefunction is normally distributed for every α , and the variance is scaled according to Eq. (8). Another issue addressed in this Letter concerns the wavepacket evolution. In particular, we found how the diffusion constant $D(\alpha)$ and the diffusion time $t_D(\alpha)$ change as a function of the time reversal symmetry breaking parameter α . As a result we were able to establish a scaling law (11) for the variance $M(t)$ of the packet.

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